Recent progress in analytical solutions of the Boussinesq nonlinear groundwater equation

3rd Brazilian Soil Physics Meeting, May 4th 2015

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May 04 2015

Lemma
UFPR

Summary

Thanks and acknowledgements

It is an honor to be here, to speak to you.

Thanks to the organizing committee, in particular to Robson Armindo, for the invitation.

This talk is largely based on Tomás Chor's MSc thesis and joint publications with Tomás Chor and Ailín Ruiz de Zárate.

2. Introduction

The Boussinesq nonlinear PDE for groundwater (for a horizontal acquifer) was derived by Boussinesq in 1903. Its assumptions are

- conservation of mass
- Darcy's law
- horizontal mean velocity (Dupuit-Forchheimer)

The 1-D version of the equation, for a horizontal aquifer, is

$$\frac{\partial h}{\partial t} = \frac{k}{n}\frac{\partial}{\partial x}\left[h\frac{\partial h}{\partial x}\right].$$

- k is the saturated hydraulic conductivity,
- n is the drainable porosity.

Meet Boussinesq

» L'existence, à une époque reculée, d'une atmosphère appréciable, ayant occasionné la diffusion des cendres sous forme de traînées;

» L'absence d'eau courante à la surface, confirmée par l'état de conservation de ces dépôts;

» L'ordre de succession de divers grands cataclysmes et celui de la consolidation des diverses parties de la surface;

» L'interprétation des recrudescences des traînées, comme le signe de petites différences d'altitude. »

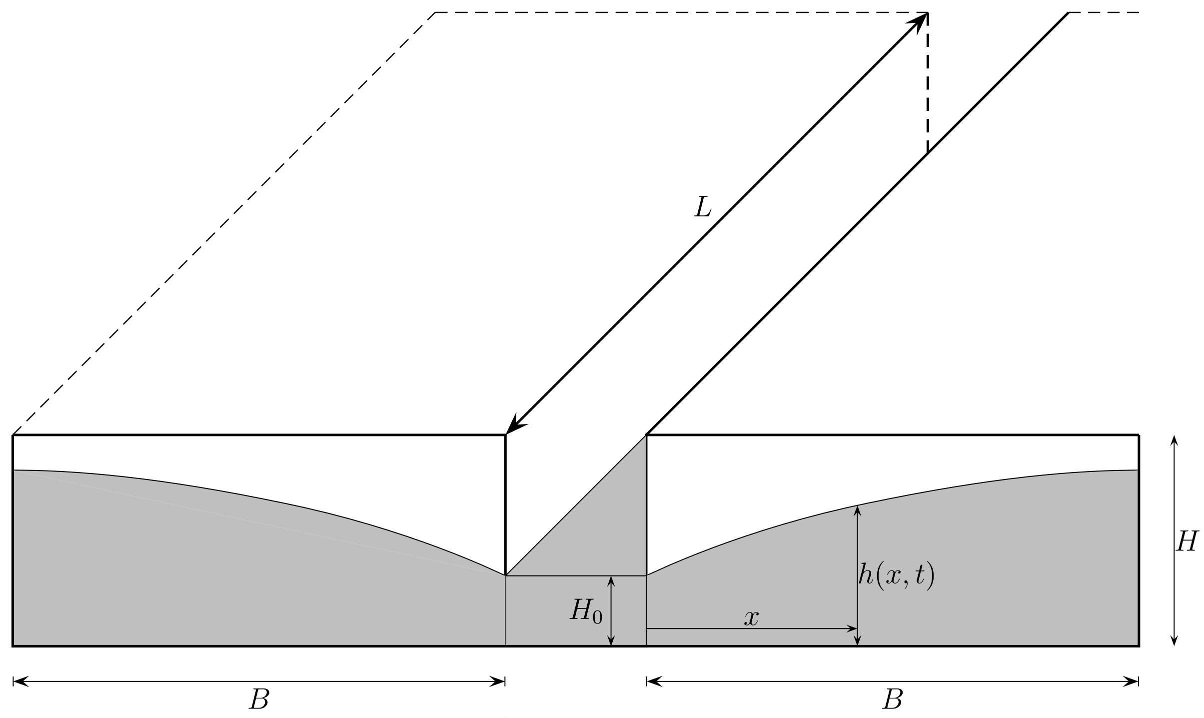
HYDRODYNAMIQUE. — *Sur le débit, en temps de sécheresse, d'une source alimentée par une nappe d'eaux d'infiltration.* Note de M. J. BOUSSINESQ.

« I. Lorsqu'un sol perméable, reposant sur un sous-sol imperméable, a ses couches inférieures imprégnées d'eau, mais ses couches supérieures soumises à la pression atmosphérique constante de l'air superposé à l'eau, et que, d'ailleurs, celle-ci, occupant les interstices des grains sablonneux ou terreux des couches inférieures, c'est-à-dire une certaine fraction μ de leur volume apparent donnée pour chaque endroit (x, y, z), se trouve animée, dans toute région un peu étendue, de lents mouvements *suivant une certaine direction générale*, susceptible de changer peu à peu avec le temps t, les interstices contigus assez bien alignés suivant cette direction générale pour permettre un écoulement appréciable, deviennent des *tubes de transpiration*. Quant à l'eau emmagasinée entre deux tubes de transpiration voisins, dans les interstices dont les ouvertures sont disposées suivant les directions perpendiculaires, et qui, à peu près immobile, complète en quelque sorte la paroi des tubes, elle a, dans le mouvement, le rôle capital de transmettre la pression d'un tube à l'autre et, par suite, de rendre solidaire dans tous l'écoulement.

» Or on sait que, le long des chemins perpendiculaires à cet écoulement, la pression est régie par la loi hydrostatique, non seulement là où le liquide est ainsi en repos entre deux tubes, mais, même, à la traversée des filets fluides de chaque tube. Abstraction faite d'anomalies locales se neutralisant en moyenne, cette pression p (que nous supposerons évaluée en hauteur de liquide), varie donc *hydrostatiquement* le long de tels chemins, dans toute l'étendue du liquide filtré par le sol. C'est dire que la *hauteur* φ *de charge*, somme, en chaque point (x, y, z), de p et de l'*altitude* z du point

Why is it important?

The equation, although it involves some approximations, is a very good model for the drought flow of an aquifer into free-surface streams, such as rivers and drainage trenches.



Why is it important?

- As such, it can predict the outflow from an aquifer accurately [Ibrahim and Brutsaert, 1965], as well as predict the free surface when $B/H > 4$ [Verma and Brutsaert, 1971, p. 1227]

- Various solutions of the Boussinesq equation can be used together to solve an "inverse problem": from the outflow hydrograph, obtain the physical and geometrical parameters of the aquifer (usually, two among: k, n, B, H). This is called "Brutsaert-Nieber" recession analysis, from its seminal paper [Brutsaert and Nieber, 1977].

- Recent uses of this equation include the linking of geological and geomorphological features to hydrological behavior [Mutzner et al., 2013, Vannier et al., 2014] and the definition of good engineering practices for the robust calibration of parsimonious models [Melsen et al., 2014].

Focus

- As usual, there is a very large field of themes to investigate regarding the Boussinesq non-linear PDE.
- In this talk, I will concentrate on the particular case $B \to \infty$. The PDE can then be simplified to an ODE.
- It is this ODE, and a sequence of proposed solutions, that we will be talking about here.

Focus

- As usual, there is a very large field of themes to investigate regarding the Boussinesq non-linear PDE.
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Unfortunately, whenever we have focus on a problem, difficulties mount.

We end up spending a lot of energy on the focused problem, which is but a small part of the problem we started with.

Please, bear with us as we probe the many facets and difficulties of some solutions of the Boussinesq ODE!

3. The Boussinesq ODE, and its cousin the Blasius ODE

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We are interested in the case $B \to \infty$.
Given a Boltzmann transformation,

$$\phi = \frac{h(x,t)}{H}, \qquad \xi = \frac{x}{\sqrt{4Dt}}, \qquad D = H\frac{k}{n},$$

the Boussinesq PDE is transformed into

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\phi\frac{\mathrm{d}\phi}{\mathrm{d}\xi}\right) + 2\xi\frac{\mathrm{d}\phi}{\mathrm{d}\xi} = 0.$$

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Surprisingly, Punnis [1956] showed that, if we further put

$$\phi = \frac{\mathrm{d}f}{\mathrm{d}\eta}, \qquad \xi = \frac{1}{2}f,$$

we obtain

$$\frac{\mathrm{d}^3 f}{\mathrm{d}\eta^3} + \frac{1}{2}f\frac{\mathrm{d}^2 f}{\mathrm{d}\eta^2} = 0,$$

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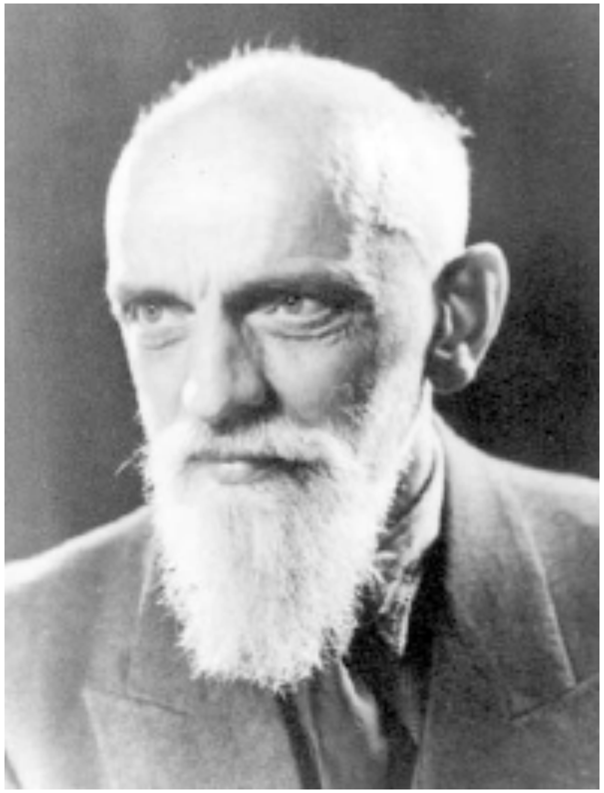
This means that in principle we can apply everything we know about Blasius' solution of f to Boussinesq's solution of ϕ. We will call this the *Punnis transformation.*

The legacy of Blasius

There is a lot to be learned from Blasius [1908]'s Boundary-Layer Theory. Remember, Blasius's advisor was Ludwig Prandtl, the creator of the Boundary-Layer concept.



Ludwig Prandtl (left) and Paul Richard Heinrich Blasius (right).

In 1950, NACA (nowadays NASA) was interested in Blasius [1908].

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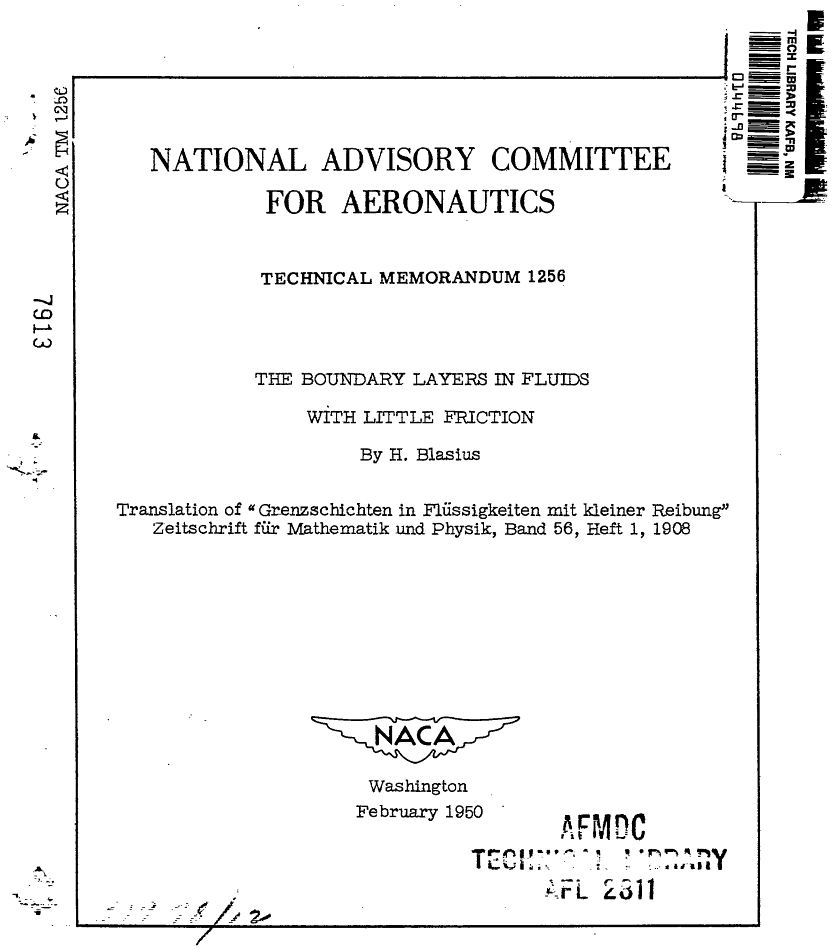

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THE BOUNDARY LAYERS IN FLUIDS
WITH LITTLE FRICTION

By H. Blasius

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NACA

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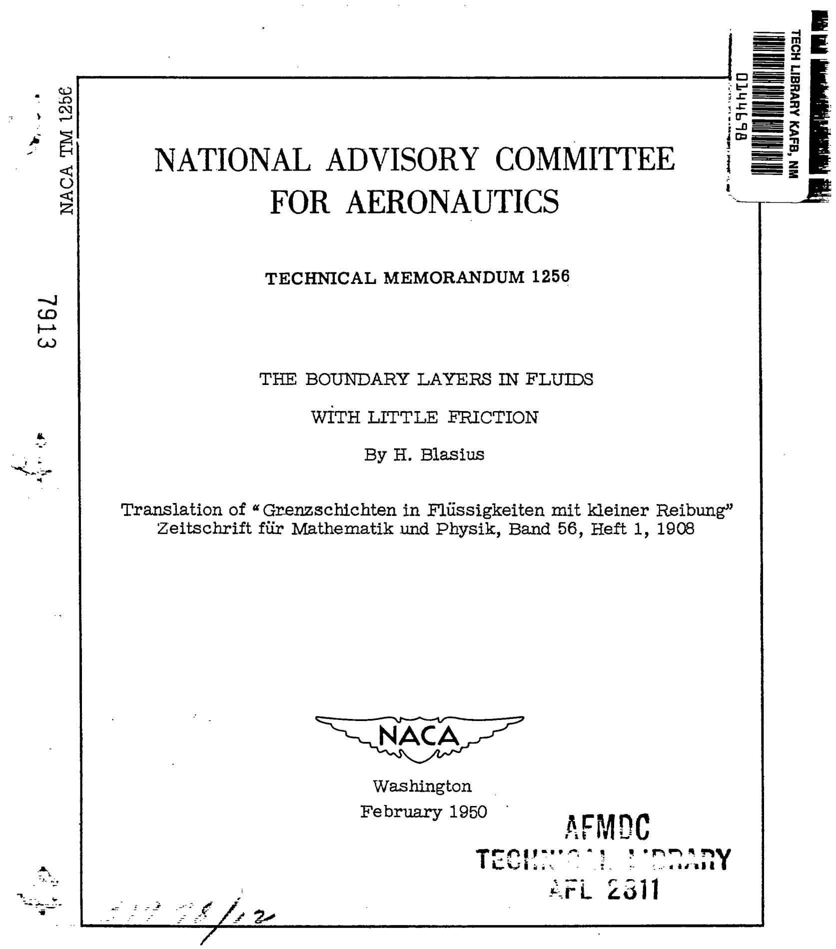
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Blasius' did not stop at deriving his equation. The main tools he used were:

- The product of two series is again a series
- A solution for f in series can be developed, but the radius of convergence around $\eta = 0$ is finite.
- Beyond the radius of convergence, Blasius implemented an asymptotic solution.

Blasius' essential mathematical tools

Series products

$$\left[\sum_{n=0}^{\infty} A_n x^n\right]\left[\sum_{m=0}^{\infty} B_m x^m\right] = \sum_{n=0}^{\infty}\left[\sum_{m=0}^{n} A_m B_{n-m}\right] x^n.$$

This allows us to seek series solutions of non-linear differential equations. Finding the radius of convergence of these solutions, however, can be quite difficult.

Asymptotic behavior

Moreover, we may try to modify the ODE by "knowing" some asymptotic behavior. In our case, if we start out from an aquifer full of water up to H at $t = 0$, and if $B = \infty$, we expect

$$\lim_{\xi\to\infty} \phi(\xi) = 1.$$

Substituting back this condition or a zero derivative in the Boussinesq ODE may result in a simpler ODE, valid for large ξ only.

4. The search for solutions

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In order for the Boltzmann transform to work, we need to collapse two Boundary Conditions into one:

$$\phi = \frac{h(x,t)}{H}, \qquad \xi = \frac{x}{\sqrt{4Dt}}, \qquad D = H\frac{k}{n},$$

$$\begin{aligned} h(x,0) &= H \\ h(\infty,t) &= H \end{aligned} \qquad \Rightarrow \qquad \phi(\infty) = 1,$$

$$h(0,t) = H_0 \qquad \Rightarrow \qquad \phi(0) = \phi_0.$$

This is a boundary-value problem. Physically, however, we are interested in the initial condition

$$\psi_0 = \phi \frac{\mathrm{d}\phi}{\mathrm{d}\xi}\bigg|_{\xi=0},$$

which gives the flow rate out of the aquifer and into the stream.

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which gives the flow rate out of the aquifer and into the stream.

- The **shooting method** consists of guessing ψ_0, solving numerically the Boussinesq ODE, checking $\phi(\infty)$ (where "∞" is just a large enough ξ), adjusting ψ_0, and going over (until $\phi(\infty) \approx 1$).
- **Töpfer's (1912) method**,

$$\phi = \lambda^{-2}\phi^*, \qquad \psi = \lambda^{-3}\psi^*, \qquad \xi = \lambda^{-1}\xi^*,$$

  avoids the iterations, but only when $\phi_0 = 0$.
- No matter what the subsequent analytical approach is, a numerical solution (*e.g.* Runge-Kutta) is needed to obtain ψ_0.

5. Solutions for $\phi_0 = 0$

Given the Blasius equation, with boundary conditions $f(0) = 0$, $f'(0) = 0$, $f'(\infty) = 1$, Blasius derived

$$f(\eta) = \sum_{n=0}^{\infty} (-1)^n a_n \eta^{3n+2}, \qquad a_n = \frac{1}{2(3n+2)(3n+1)(3n)} \sum_{k=0}^{n-1} (3k+2)(3k+1) a_k a_{n-1-k},$$

where $a_0 = \kappa/2$ and $f''(0) = \kappa = 0.33205733621519630$ [Boyd, 2008].

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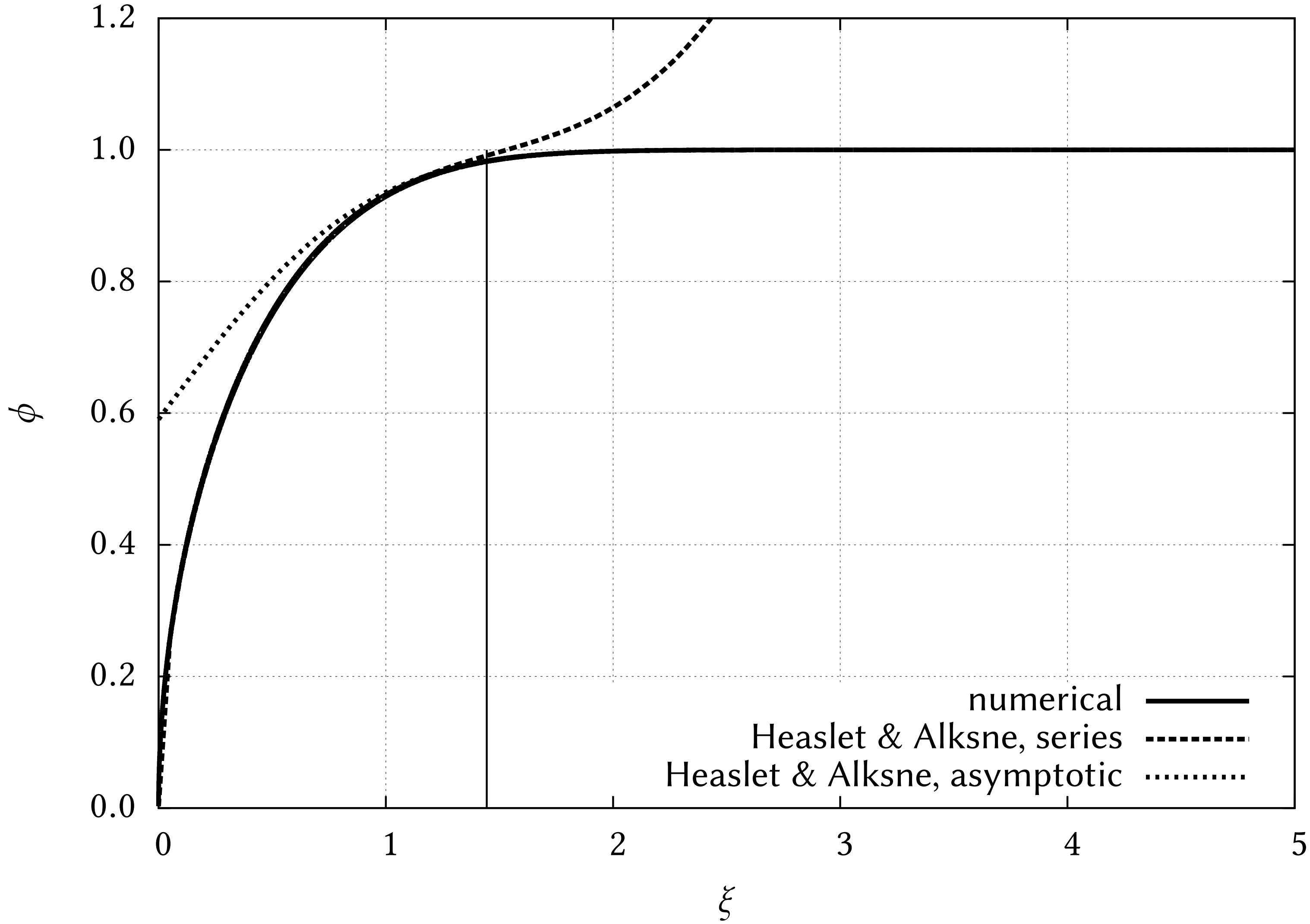
Using the Punnis transformation, both Heaslet and Alksne [1961] and Polubarinova-Kochina [1962] inverted the Blasius series and obtained a few terms of a series solution for the Boussinesq EDO; they also obtained from Blasius an asymptotic solution:

$$\phi(\xi) = 1.152\xi^{1/2} - \frac{4}{15}\xi^2 + 0.0462\xi^{7/2} - 0.00065\xi^5 + \ldots,$$

$$\phi(\xi) \approx 1 - 0.231\sqrt{\pi}\,\mathrm{erfc}(\xi).$$

Heaslet and Alksne [1961]'s results were already very good



Improvements: Hogarth and Parlange [1999]

The next step was provided by Hogarth and Parlange [1999]: they replaced the Heaslet& Alksne–Polubarinova-Kochina series solution with a Padé approximation obtained from that series, and also improved the asymptotic solution:

$$\phi(\xi) = 1.15249\xi^{1/2} - \frac{4/15\xi^2}{1 + 0.17355\xi^{3/2} + 0.02768\xi^3}, \qquad \xi \le 1.3,$$

$$\phi(\xi) = 1 - 0.41387\,\mathrm{erfc}\left(\frac{\xi}{1 + 0.058375\xi^3\exp(-\xi^2)}\right), \qquad \xi > 1.3.$$

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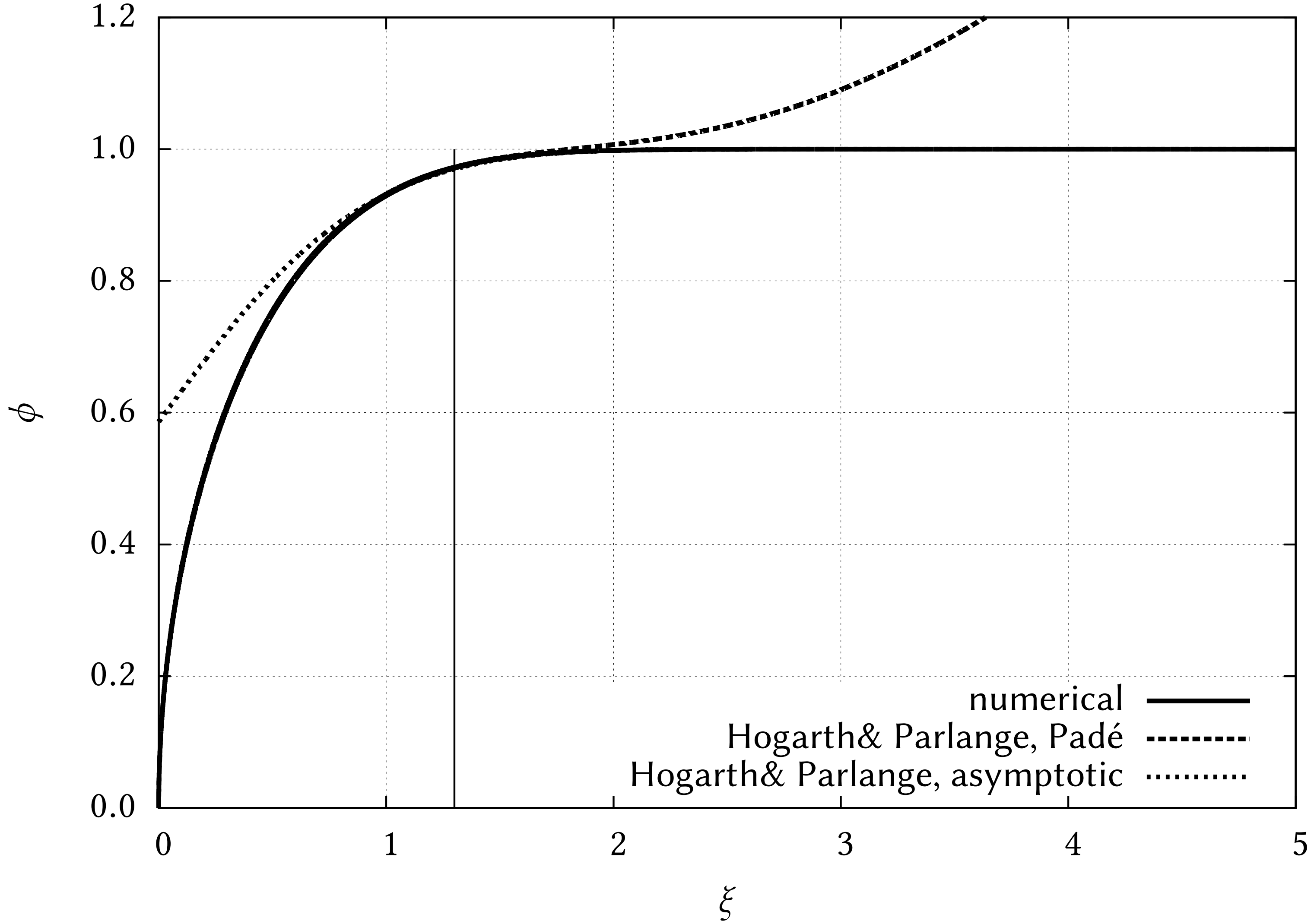
So far, everything was being obtained on the basis of Blasius' results alone!

Hogarth and Parlange [1999]'s results were more accurate

(but visually not too different)



Direct series solution of the Boussinesq ODE

We approached the problem again in Tomás Chor's MSc thesis, but we searched directly for a series solution of the Boussinesq ODE:

$$\phi(\xi) = \sum_{n=0}^{\infty} a_n \xi^{(3n+1)/2}.$$

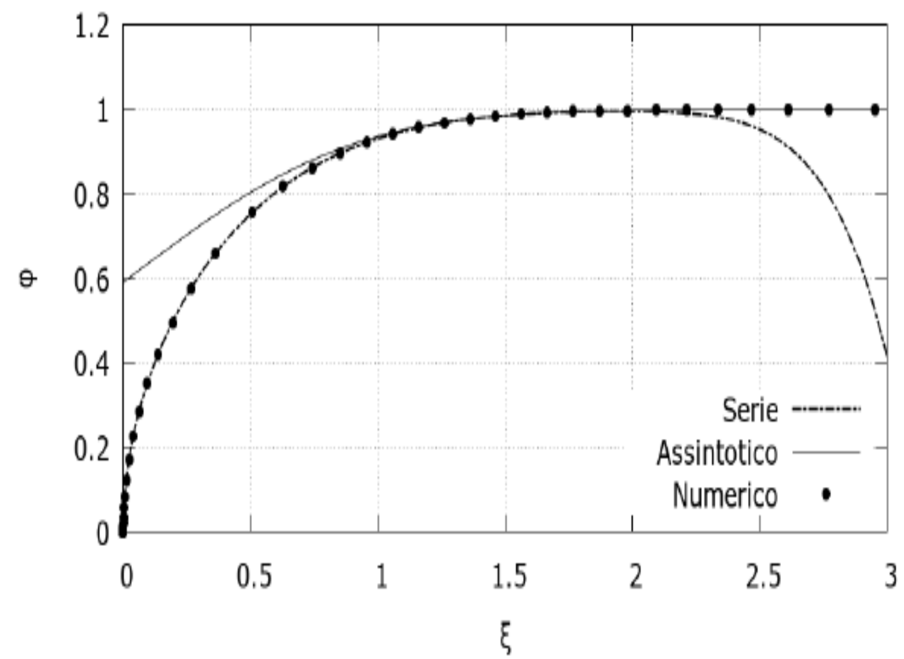
We first used symbolic algebra to compute the first n values of the series:

```
M : 50 ;
a[0]: 1.15248832742929 ;
fi : sum(a[n]*x^((3*n+1)/2),n,0,M) ;
eq : expand( diff(fi*diff(fi,x),x) + 2*x*diff(fi,x) ) ;
cond : [] ;
file_output_append : true ;
with_stdout("se_xi.txt",print(string(float(a[0]))));
for n : 0 thru M-1  do (
   eqqu : ev(coeff(eq,x,(3*n+1)/2),cond) ,
   this : solve(eqqu,a[n+1]),
   cond : append(cond,this),
   this : first(this),
   a[n+1] : rhs(this),
   with_stdout("se_xi.txt",print(float(a[n+1])))
);
```

Results with symbolic algebra

We can go as far as we want with the a_n's:

a_0	$+1,15248832742929 \times 10^{+0}$
a_1	$-2,6666666666667 \times 10^{-1}$
a_2	$+4,6276679827463 \times 10^{+0}$
a_3	$-6,489488169252842 5 \times 10^{-4}$
a_4	$-9,4517828664332276 \times 10^{-4}$
a_5	$-2,5400784492116708 \times 10^{-5}$
a_6	$+3,5810599703874529 \times 10^{-5}$
a_7	$+3,8844564686017651 \times 10^{-6}$
a_8	$-1,4163383082670557 \times 10^{-6}$
a_9	$-3,2740710683167304 \times 10^{-7}$
a_{10}	$+4,5534502970990704 \times 10^{-8}$



From Chor et al. [2013b]

With hard algebra

Next, results were derived analytically [Chor et al., 2013a]:

$$a_{n+1} = -\frac{1}{a_0(3n+5)}\left(\frac{4\,(3n+1)\,a_n}{3n+3} + \frac{3n+5}{2}\sum_{k=1}^{n} a_k a_{n-k+1}\right),$$

a_0	$\sqrt{2\psi_0}$
a_1	$-\frac{4}{15}$
a_2	$-\frac{1}{6a_0}(2a_1 + 3a_1^2)$
a_3	$-\frac{(99a_1+28)a_2}{99a_0}$
a_4	$-\frac{(42a_1+10)a_3+21a_2^2}{42a_0}$
$\vdots$	$\vdots$

Some limitations are inherited from the Blasius' series solution:

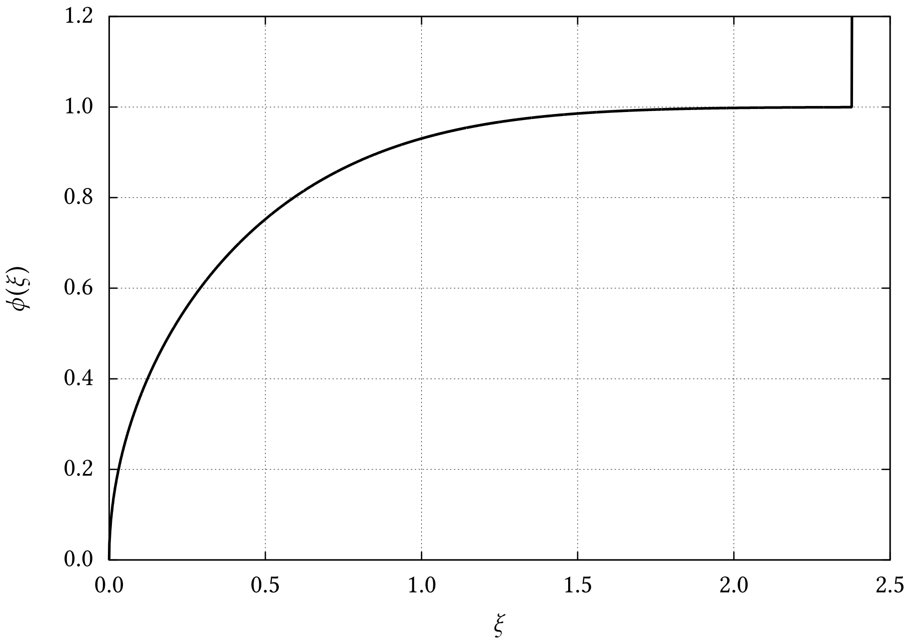
- No equation for the general term (a reccurence relation instead): no easy way to calculate the radius of convergence.
- Dependence on the first term. In our case,

  $$\psi_0 = \phi\frac{d\phi}{d\xi}\bigg|_{\xi=0}$$

  is a key parameter.

Still, the radius of convergence was not known

Series solution calculated with **8700** terms:



To obtain an accurate estimate of the radius of convergence, we changed variables ($\zeta = \xi^{1/2}$), obtained a new equation,

$$\frac{\mathrm{d}}{\mathrm{d}\zeta}\left[\frac{\Phi}{\zeta}\frac{\mathrm{d}\Phi}{\mathrm{d}\zeta}\right] + 4\zeta^2\frac{\mathrm{d}\Phi}{\mathrm{d}\zeta} = 0,$$

and a standard power series

$$\Phi = \sum_{n=0}^{\infty} a_n \zeta^{3n+1}.$$

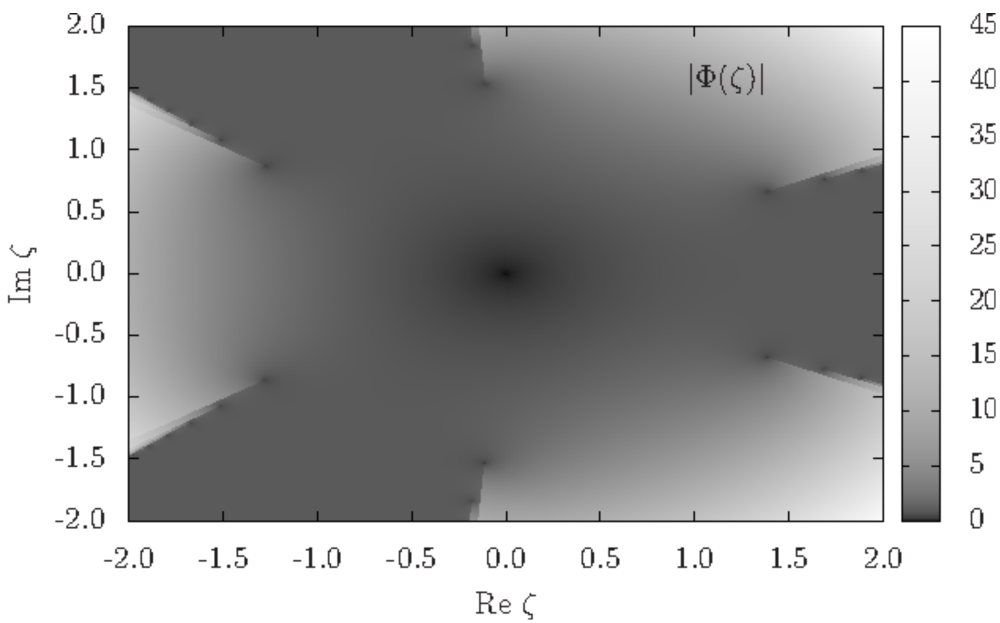
$\Phi(\zeta)$ is an analytic function within a circle around the origin in the **complex ζ-plane**. The circle's radius is the radius of convergence. Within that radius, from Cauchy's Theorem,

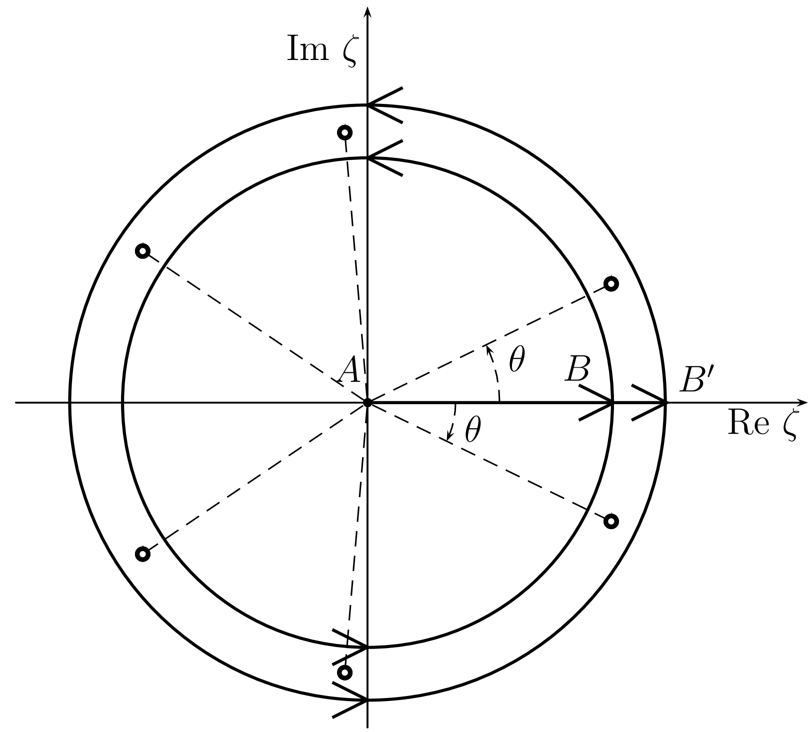
$$\oint_C \Phi(\zeta)\,\mathrm{d}\zeta = 0.$$

Lemma UFPR

The radius of convergence

$$R \approx 2.3757445$$





All integrals are line integrals using the Runge-Kutta-Cash-Karp method in the complex plane. The method provides error estimates which are used to identify non-zero line integrals. Six singularities of the complex Boussinesq function $\Phi(\zeta)$ are clearly seen, along with branch cuts emanating from them.

Chor et al. [2013a] improvements

Having the full series now, we extended the Padé approximations

$$P_N(\xi) = \xi^{1/2} \frac{\sum_{n=0}^{N} A_n \xi^{\frac{3n}{2}}}{\sum_{n=0}^{N} B_n \xi^{\frac{3n}{2}}}.$$

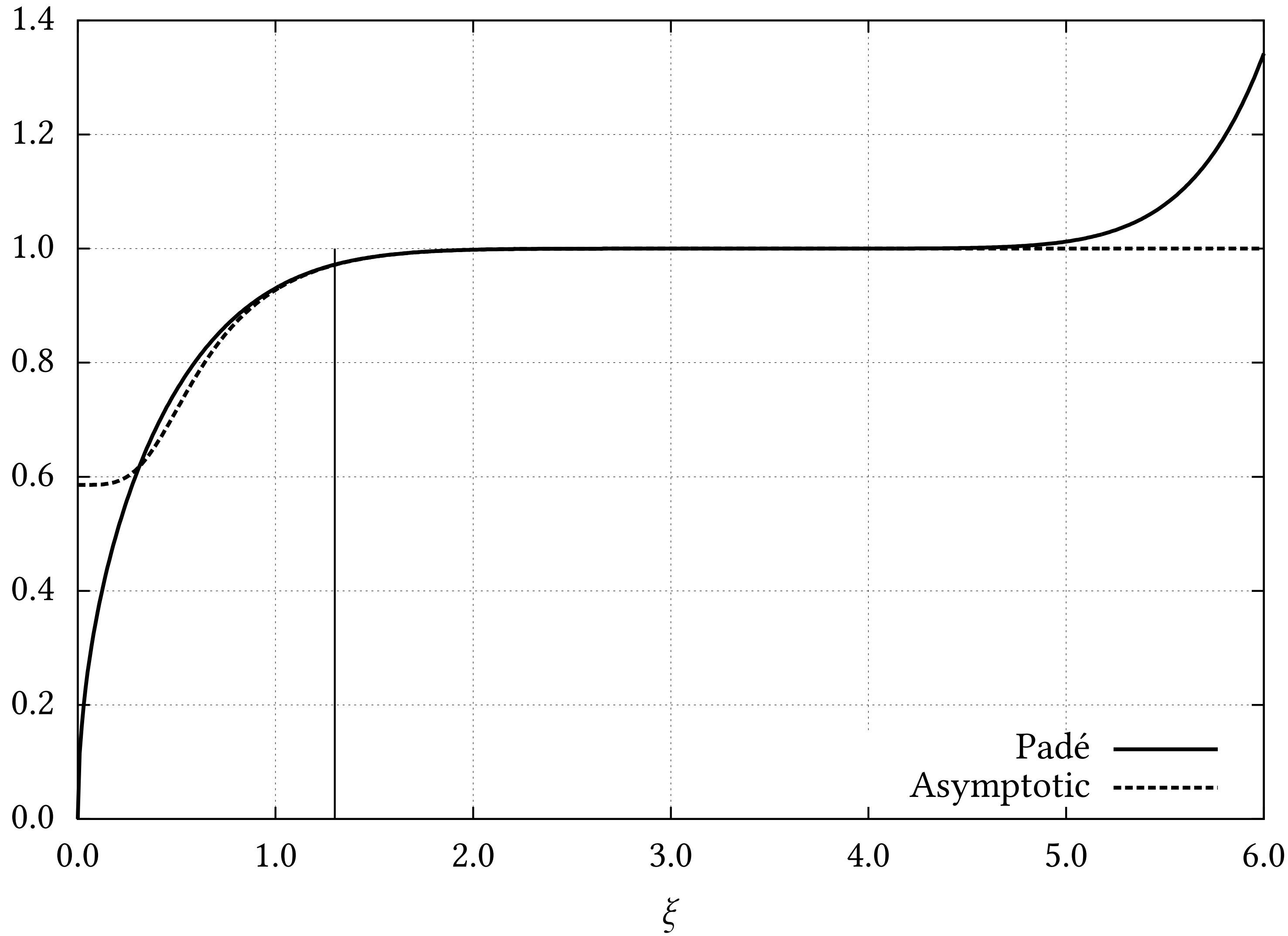
Best results were obtained for $N = 200$, and a more modest $N = 20$ was tabulated. With $N = 20$, the Padé approximation extends the radius of convergence to ≈ 3.3.

The same asymptotic solution of Hogarth and Parlange [1999],

$$\phi(\xi) \sim 1 - A\sqrt{\pi}\,\mathrm{erfc}\left(\frac{\xi}{1 + \frac{A}{2\xi^3}\exp(-\xi^2)}\right)$$

was also obtained, but with $A = 0.2337276186438419$.

Best results (But probably overkill)



6. Solutions for $\phi_0 \neq 0$

We now look at the boundary condition $0 < \phi_0 < 1$. As before, $\phi(\infty) = 1$.

Differently from the case $\phi_0 = 0$, the derivative $\mathrm{d}\phi/\mathrm{d}\xi$ is not singular at $\xi = 0$, and this allows one to seek a solution in terms of a regular Taylor series [Dias et al., 2014]:

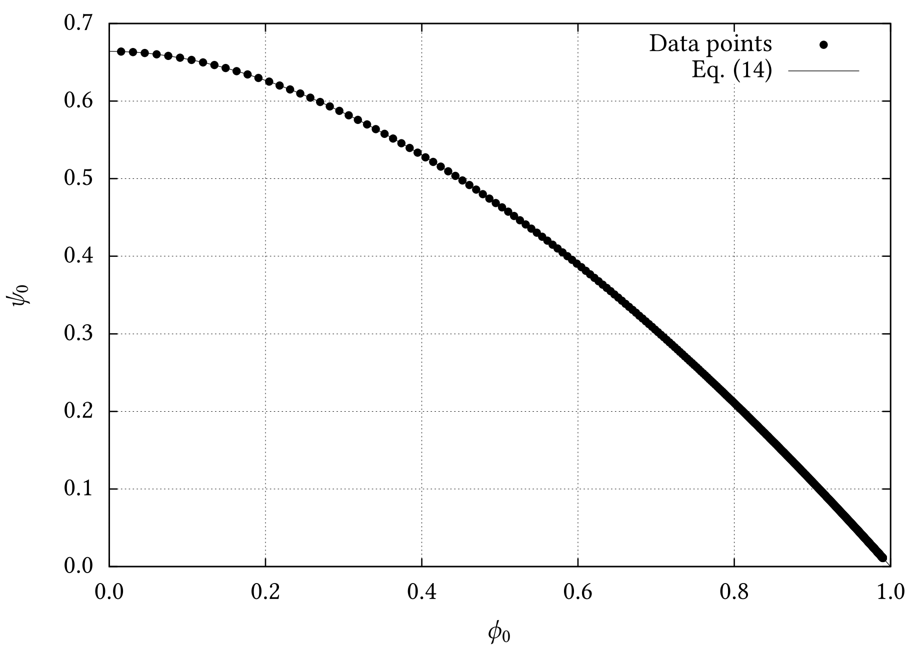
$$\phi(\xi) = \sum_{n=0}^{\infty} a_n \xi^n,$$

$$a_0 = \phi_0, \qquad a_1 = \frac{\psi_0}{a_0},$$

$$a_{n+1} = -\frac{1}{a_0(n+1)}\left[\frac{2(n-1)}{n} a_{n-1} + \sum_{k=1}^{n} (n-k+1) a_k a_{n-k+1}\right].$$

A practical approach for the initial condition

As before, the value of ψ_0 is of engineering interest, but cannot be obtained (so far) by purely analytical means. Töpfer's method can no longer be applied to obtain ψ_0 (numerically) in one pass, but it can help to generate a large number of numerical solutions [see Dias et al., 2014]. Then, we fitted an empirical curve:



$$\psi_0(\phi_0) \approx (\Psi_0^d + a\phi_0^b)^{\frac{1}{d}}(1 - \phi_0^c)(1 + f\phi_0^g)^e,$$

$$\Psi_0 \approx 0.66411467.$$

with $a = 0.733841$, $b = 0.999223$, $c = 0.98359$, $d = 2.94568$, $e = 0.186587$, $f = 0.966673$, and $g = 0.93347$.

A new asymptotic solution

When $\xi \to \infty$, $\phi \to 1$. Substituting this into the Boussinesq ODE:

$$\phi\phi'' + \phi'\phi' + 2\xi\phi' = 0 \qquad \Rightarrow$$
$$\phi_a'' + \phi_a'\phi_a' + 2\xi\phi_a' = 0$$

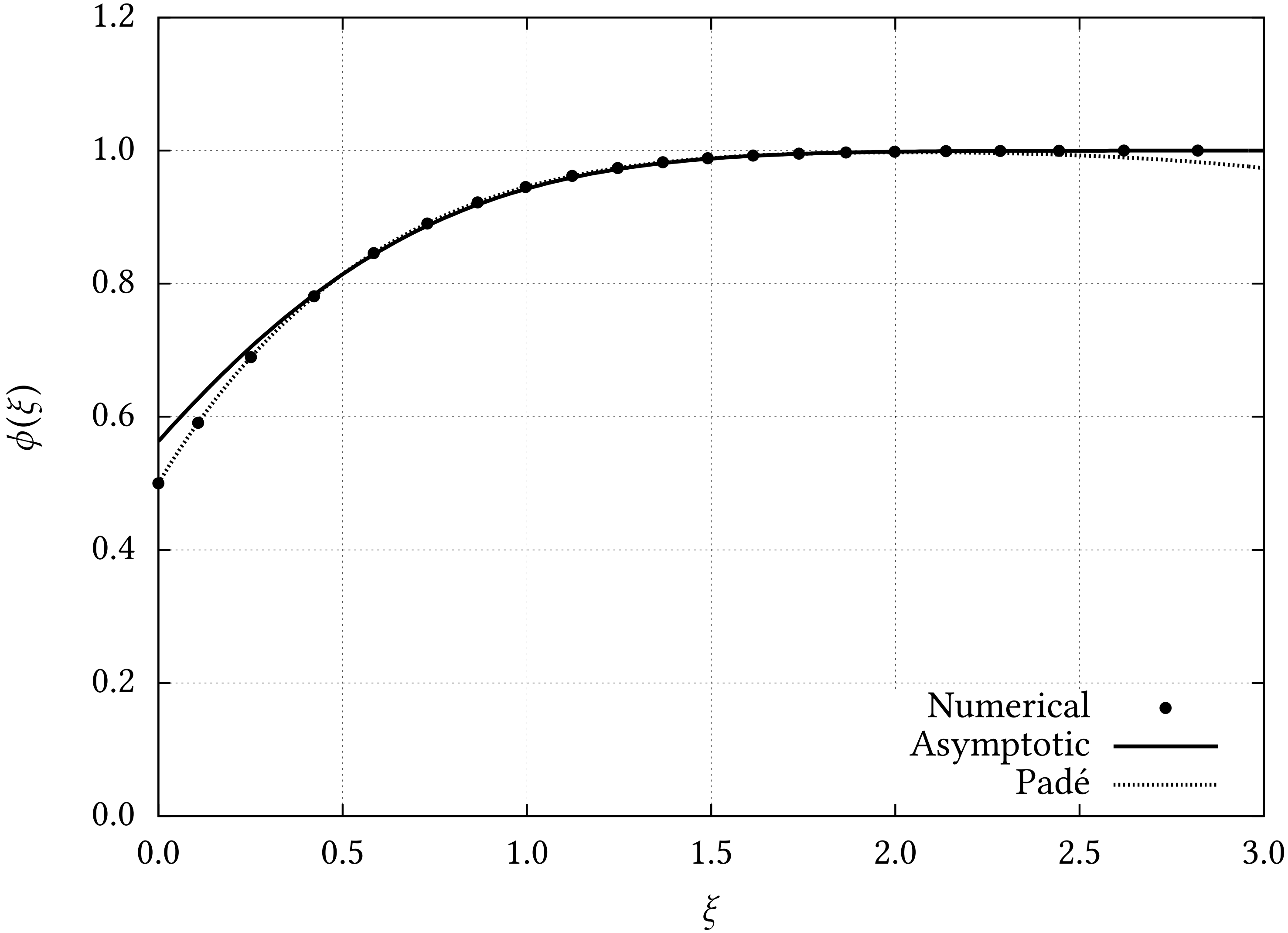
(where a is for "asymptotic"). The latter is a Bernoulli equation in ϕ_a'! Solving,

$$\phi_a(\xi, C) = \ln\left(\frac{\operatorname{erf}(\xi) + C}{1 + C}\right) + 1.$$

For each ϕ_0 there is an optimum C:

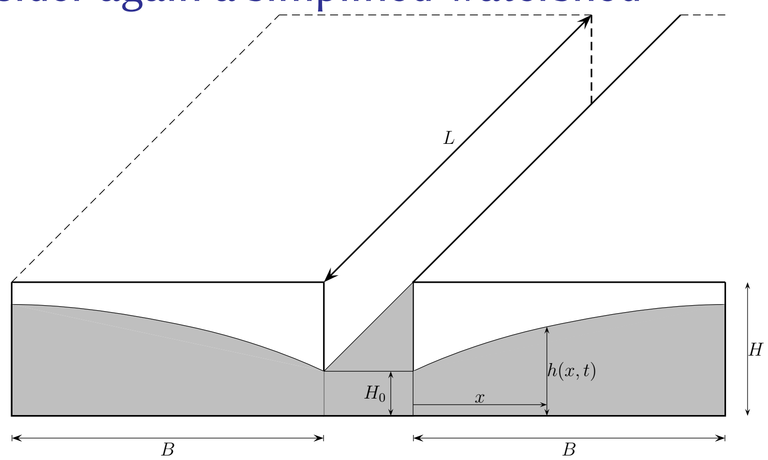
ϕ_0	C
0.1	.863074
0.2	1.01683
0.3	1.20578
0.4	1.46309
0.5	1.82544
0.6	2.39555
0.7	3.22508
0.8	4.87223
0.9	9.84770

Results from Dias et al. [2014] for $\phi_0 = 0.5$



7. An application to hydrology

Consider again a simplified watershed:



In the dimensionless variables

$$\mathrm{x} = \frac{x}{B}, \qquad \tau = \frac{kH}{nB^2}t,$$

the Boussinesq PDE is

$$\frac{\partial \phi}{\partial \tau} = \frac{\partial}{\partial \mathrm{x}}\left[\phi \frac{\partial \phi}{\partial \mathrm{x}}\right],$$

$$\phi(\mathrm{x},0) = 1, \qquad \phi(0,t) = \phi_0, \qquad \frac{\partial \phi}{\partial \mathrm{x}}(1,t) = 0.$$

At an early time the aquifer "looks" infinite along x, and the dimensionless outflow is

$$\chi(\tau) = \left[\phi \frac{\partial \phi}{\partial \mathrm{x}}\right](0,\tau) = \left[\phi \frac{\mathrm{d}\phi}{\mathrm{d}\xi}\right]_0 \frac{1}{2}\tau^{-1/2} = \frac{\psi_0}{2}\tau^{-1/2}.$$

For late times, a linearized equation can be derived [Boussinesq, 1904],

$$\frac{\partial \phi}{\partial \tau} = p\frac{\partial^2 \phi}{\partial \mathrm{x}^2}$$

whose solution reduces to

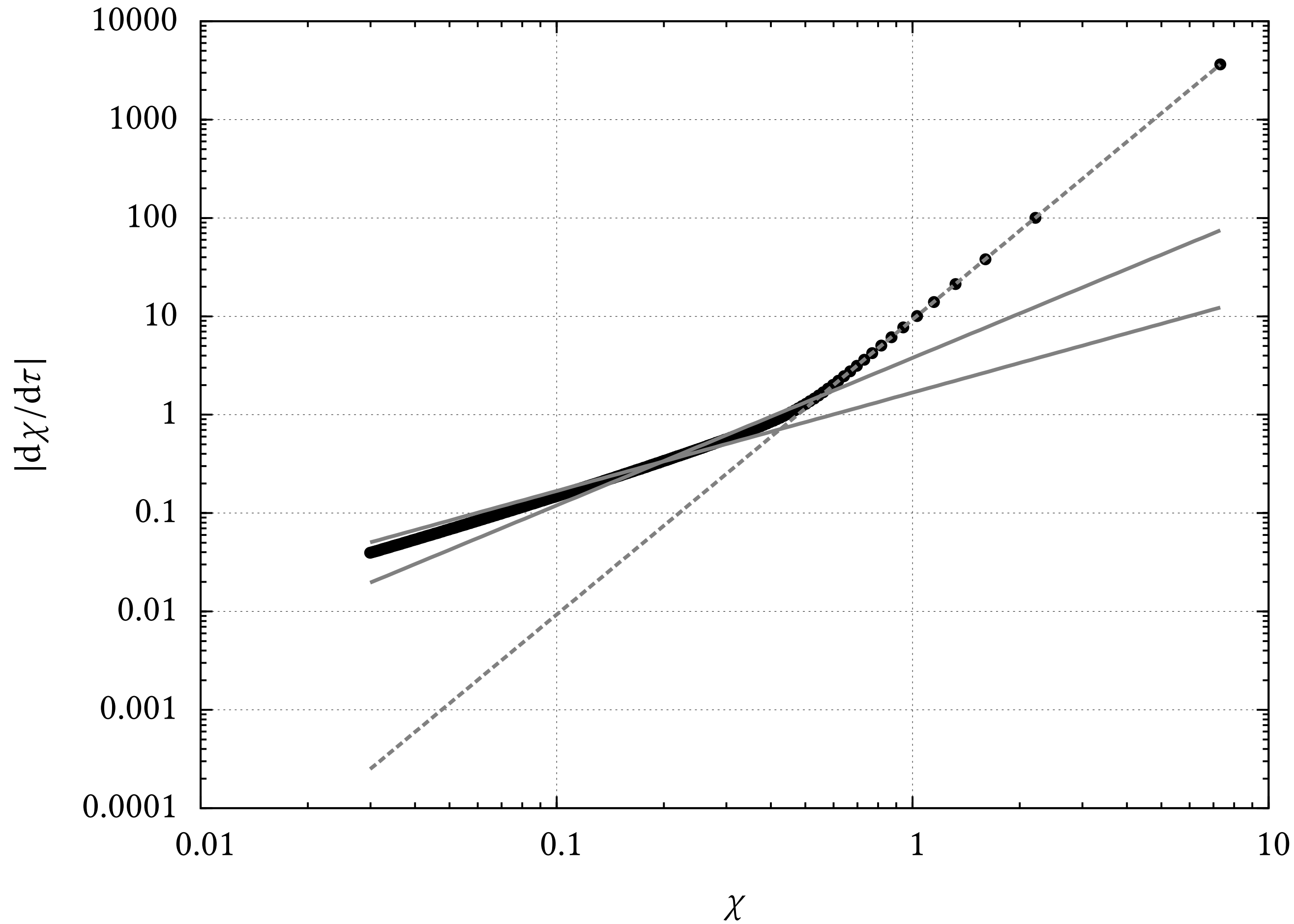
$$\chi(\tau) = p \exp\left(\frac{-\pi^2 p}{4}\tau\right).$$

Hence,

$$\frac{\mathrm{d}\chi}{\mathrm{d}\tau} = \alpha \chi^{\beta}$$

Brutsaert-Nieber recession analysis

α_1 and α_2 are analytically related to k and n. $\beta_1 = 3$ and $\beta_2 = 1$ are predicted from the analytical solutions.



Retrieving n and k

From Chor and Dias [2014]:

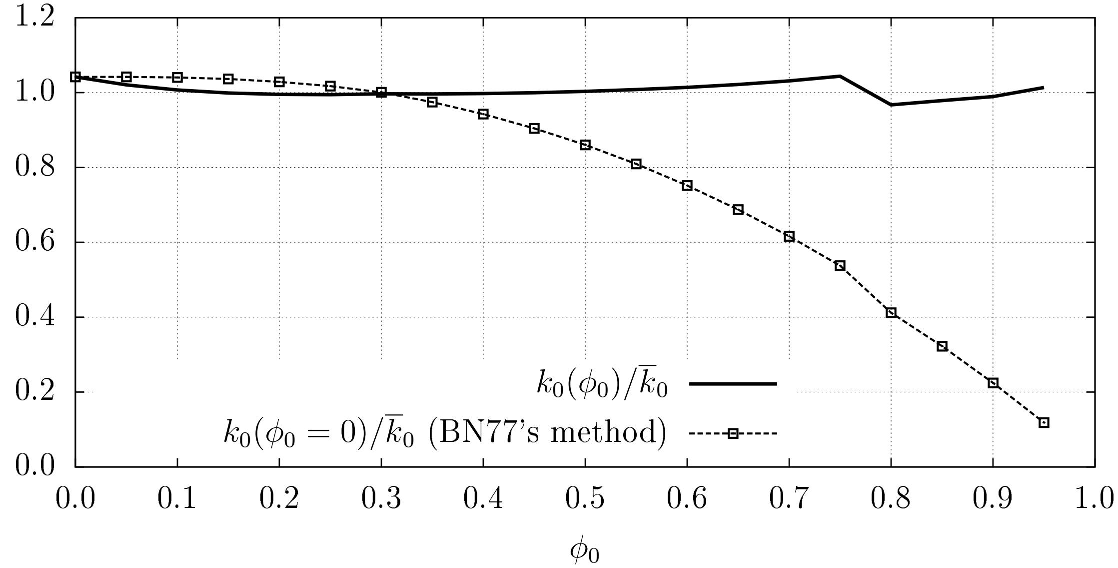
$$n = \left(\frac{p}{2}\right)^{1/2} \frac{\pi}{H\psi_0 A} \left(\alpha_2\alpha_1\right)^{-1/2},$$

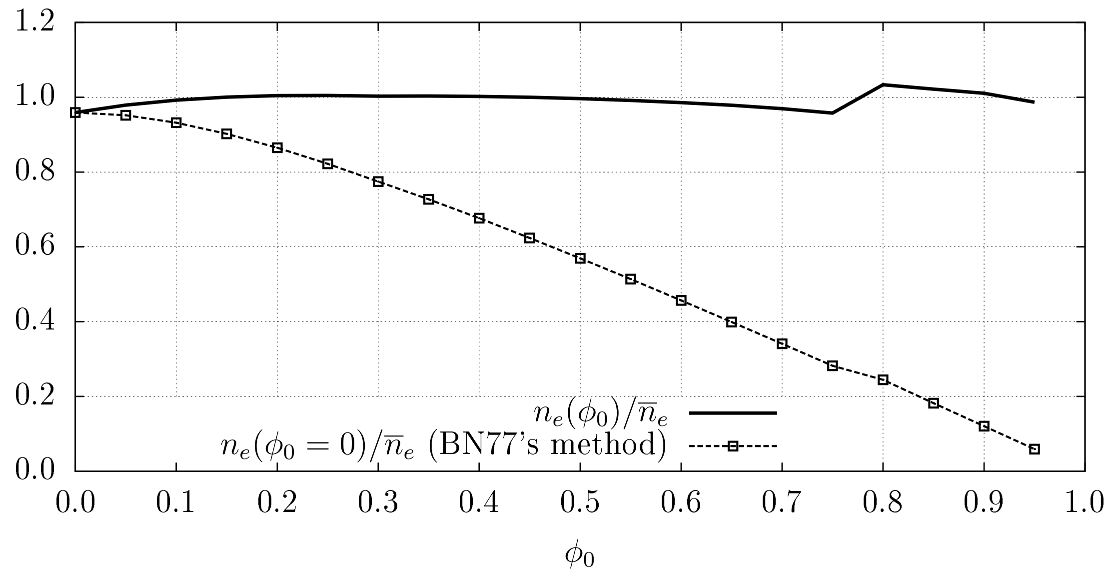
$$k = \frac{A}{\sqrt{2p}H^2L^2\pi\psi_0} \left(\frac{\alpha_2}{\alpha_1}\right)^{1/2}.$$

All of previous research (including analyses with real watersheds) has used the value of ψ_0 associated with $\phi_0 = 0$.

However, from Dias et al. [2014], we now have the function $\psi_0(\phi_0)$.

Effect of changing ψ_0.





8. Conclusions

You may know more about this talk in:

Tomás Luís Guimarães Chor, Nelson Luís Dias, and Ailin Ruiz de Zarate. Solução em série da equação de boussinesq para fluxo subterrâneo utilizando computação simbólica. In *Anais, XX Simpósio Brasileiro de Recursos hídricos*, Bento Gonçalves, RS, 2013b

T. Chor, N. L. Dias, and A. R. de Zárate. An exact series and improved numerical and approximate solutions for the boussinesq equation. *Water Resour. Res.*, 49:7380–7387, 2013a. doi: 10.1002/wrcr.20543

N. L. Dias, T. Chor, and A. R. de Zárate. A semi-analytical solution for the boussinesq equation with non-homogeneous constant boundary conditions. *Water Resour. Res.*, 50 (8):6549–6556, 2014. doi: 10.1002/2014WR015437

T. L. Chor and N. L. Dias. A simple generalization of the brutsaert and nieber analysis. *Hydrology and Earth System Sciences Discussions (Online)*, 11:12519–12530, 2014

Closing remarks

- To this day, the analytical (series and asymptotic) solutions of Blasius are a powerful tool, not only for Boundary-Layer Theory, but for a completely unrelated problem, the Boussinesq non-linear equation.

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- We must recognize that the first strides (Blasius, Polubarinova-Kochina, Heaslet and Alksne) were the largest, but the recent improvements promise a much wider scope of applications and unprecedented (maybe not needed in Engineering) accuracy.
- I leave you with the most accurate (31 digits) estimate of Blasius's constant for the shear stress in a laminar boundary-layer, from Chor et al. [2013a]:

$$\kappa = 0.3320573362151962989371800593389\mathbf{2}.$$

Many thanks

…for your attention!

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Questions?

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